Bull. Sci. math.
1998, 122, p. 67-82.

ON HOLOMORPHIC FACTORIZATION AND MEROMORPHIC SELECTORS

BY

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ABSTRACT. – We show that, generically, a smooth Lorentzian 2×2 matrix valued function A, defined on the unit circle and possessing a so called exterior meromorphic section, can be extented into the unit disk in a way that admits a non-degenerate holomorphic factorization P^*P, where the adjoint is taken with respect to the constant Lorentz matrix. This has implications for meromorphic approximation and Levi-flat surfaces with generalized disk fibers. © Elsevier, Paris.

1. Introduction

When $a > 0$ is a smooth function defined on the unit circle T, then it is well known that there exists a nonzero holomorphic function p in the unit disk Δ and smooth up to the boundary such that

$$\bar{p}ap = 1 \quad \text{on } T.$$

(*) Manuscript presented by Paul MALLIAVIN, received in September 1994.
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To be precise, let u be the harmonic extension of $-\frac{1}{2}\log a$, and let v be a harmonic conjugate to u; then $p = \exp(u + iv)$ will do, since in suitable regularity classes, v will be as regular as u (see [7] § 1.6.11). This is perhaps the simplest case of holomorphic factorization. If we interpret a as being a metric on the trivial line bundle $T \times \mathbf{C}$, we thus get an extension of the metric through $a = \bar{p}^{-1}p^{-1}$; considered as a metric on the trivial holomorphic line bundle $\Delta \times \mathbf{C}$, its curvature is zero,

$$\Delta \log a = 0,$$

since p is holomorphic.

When we have a smooth positive-definite matrix-valued function A on T, it is possible to find a corresponding matrix-valued function P, holomorphic and invertible in Δ and smooth up to the boundary, such that

$$P^H AP = I \quad \text{on } T,$$

where I is the identity matrix. This is implied already in the work of WIENER and MASANI [8]; they even considered boundary matrices A satisfying $A \in L^1(T)$ and log det $A \in L^1(T)$, and obtained a holomorphic Q with boundary values $Q \in L^2(T)$ satisfying $Q^H Q = A$ almost everywhere on T. As above, the extension into Δ, $A = P^{-H}P^{-1}$, will have vanishing curvature,

$$\frac{\partial}{\partial \bar{z}}\left(A^{-1}\frac{\partial A}{\partial z}\right) = 0,$$

since P is holomorphic (for a discussion of the curvature of a holomorphic vector bundle, consult e.g. KOBAYASHI ([5] Ch.I, § 4)).

In this paper, we investigate the solvability of a corresponding problem when A is a Lorentzian matrix-valued function on T: find holomorphic P, invertible everywhere in Δ and smooth up to the boundary, such that

$$P^H AP = L \quad \text{on } T.$$

Part of the motivation for this study is the intimate connection to Levi-flat surfaces and holomorphic (meromorphic) functions with controlled boundary behavior (see Section 2).

In a famous paper by ADAMYAN, AROV and KREIN [1], the search for holomorphic functions that minimize the $L^\infty(T)$ distance to a given $f \in L^\infty(T)$ leads to the investigation of infinite Hankel matrices. In this paper, it is natural to discuss the spectrum of a certain compact operator and $L^\infty(T)$ approximation by meromorphic functions with a fixed number of poles.

In Section 2, apart from introducing some notation, we discuss briefly the connection between Lorentzian matrices and Levi-flat surfaces.

In Section 3, we formulate the problem in a precise way, make some useful reductions, discuss necessary and sufficient conditions for solutions, and formulate and prove the main result, Theorem 3.5. There we also discuss an example.

2. Preliminaries

We call a 2×2 matrix Lorentzian if it is Hermitian with one positive and one negative eigenvalue, and denote by

$$L = (\ell_{ij}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

the standard Lorentzian matrix of type 2×2. Of course, a matrix A is Lorentzian if and only if there is a matrix P such that $P^H AP = L$, where $P^H = \bar{P}^T$ denotes the Hermitian transpose.

Let $T = \{z \in \mathbf{C};\ |z| = 1\}$ be the unit circle, and $\Delta = \{z \in \mathbf{C};\ |z| < 1\}$ the unit disk. Given smooth functions $c : T \to \mathbf{C}$ and $r : T \to \mathbf{R}_+$ (the center and radius functions, respectively), we consider the disk-valued function

$$T \ni z \mapsto \{(z, w) \in T \times \mathbf{C};\ |w - c(z)| \le r(z)\}. \tag{2.1}$$

We are interested in finding smooth extensions c and r into Δ such that the set

$$\{(z, w) \in \Delta \times \mathbf{C};\ |w - c(z)| = r(z)\} \tag{2.2}$$

defines a Levi-flat surface, and hence is foliated by analytic disks. If such extensions can be found, we obtain a lot of bounded holomorphic functions h in Δ, smooth up to the boundary and approximating the center function in the sense that

$$|h(z) - c(z)| \le r(z) \quad \text{on } T \tag{2.3}$$

(see the works by Barrett [3] and Berndtsson [4]).

If we identify $\mathbf{C}$ with the set $U_0 = \{[(w_0, w_1)] \in \mathbf{P};\ w_0 \neq 0\}$ in projective space ($w = w_1/w_0$), there is a unique Lorentzian matrix-valued function A on T such that $\det A = -1$ and such that

$$\left\{(z, [w_0, w_1]) \in T \times \mathbf{P}; [\bar{w}_0\, \bar{w}_1]\, A(z) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \leq 0\right\}$$

reproduces (2.1); in fact, putting $B = r^{-2}$,

$$A = \frac{1}{\sqrt{B}} \begin{bmatrix} \bar{c}Bc - 1 & -\bar{c}B \\ -Bc & B \end{bmatrix}, \tag{2.4}$$

as is indicated in [2], § 2.3.

The existence of the Levi-flat surface (2.2) implies the existence of a holomorphic matrix-valued function P such that, letting A denote the extension into Δ of A according to (2.4),

$$P^H A P = L. \tag{2.5}$$

Conversely, if we have a P that is a holomorphic matrix-valued function, everywhere invertible and satisfying (2.5) on T, then $P^{-H} L P^{-1}$ is a natural extension of A into Δ; it is in fact the only extension with zero curvature. However, this extension need not correspond to disk fibers (functions c and r), so, in general, we must also allow fibers that are half planes and exteriors of circles (i.e., all sets of $\mathbf{P}$ that can be obtained by Möbius transformations of the unit disk). The generalized Levi-flat surface

$$\left\{(z, [w_0, w_1]) \in \Delta \times \mathbf{P};\ [\bar{w}_0\, \bar{w}_1]\, A(z) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0\right\}$$

will now be foliated by meromorphic disks, and the best we can do in this situation is to find mermorphic functions (instead of holomorphic ones) that approximate the center function c in the sense of (2.3).

3. The problem

3.1. Formulation.. – Suppose now that we have a smooth Lorentzian matrix-valued function A defined on the unit circle. When is it possible to find a matrix-valued function P, holomorphic in Δ and smooth up to the boundary, such that $P^H A P = L$ on T? The main result of this paper is that it is generically true for a certain subclass $\mathcal{L}$ of Lorentzian matrix-valued functions; we will however formulate our result in a more geometric way.

Let $H(\Delta)$ denote the set of functions that are holomorphic in Δ, and $C^{k,\alpha}(\bar{\Delta})$ those functions whose derivatives of order k are uniformly Hölder continuous with exponent α. If $k \geq 2$ is an integer and $0 < \alpha < 1$, we consider the trivial $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ vector bundle $\bar{\Delta} \times \mathbf{C}^2$ of complex dimension two over the closed unit disk $\bar{\Delta}$; the transition functions are thus elements of $GL(2, H(\Delta) \cap C^{k,\alpha}(\bar{\Delta}))$, the set of invertible 2×2-matrices with elements in $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$. Then $\mathcal{L}$ is defined to be the set of Lorentzian metrics A defined on the trivial $C^{k,\alpha}(T)$-bundle $T \times \mathbf{C}^2$ and having the following properties:

1° $A \in C^{k,\alpha}(T)$.

2° There exists a nonvanishing $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$-section m_e of $\bar{\Delta} \times \mathbf{C}^2$ with

$$\langle m_e, m_e \rangle_A > 0 \quad \text{on } T$$

(m_e will be called an exterior section, and may be considered as a section of $\bar{\Delta} \times \mathbf{P}$).

We introduce the notation $\mathcal{M}_n$, $n = 0, 1, 2, \ldots$, for the set of nonvanishing $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$-sections that intersect the exterior section m_e precisely n times in Δ and nowhere on T, counted with multiplicities and considered as sections of $\bar{\Delta} \times \mathbf{P}$. Moreover, we let $\mathcal{S}_n = \{m \in \mathcal{M}_n;\ \langle m, m \rangle_A \leq 0 \text{ on } T\}$, and $\mathcal{S}_{<n} = \cup_{k<n} \mathcal{S}_k$. The members of $\mathcal{S}_n$ will be called *selectors;* and when we think of m_e as the infinity section, n is the number of poles of the selectors in $\mathcal{S}_n$.

THEOREM 3.1. – *The following is generically true for $A \in \mathcal{L}$:*

There exists a global $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ frame field $[s_1\ s_2]$ on $\bar{\Delta} \times \mathbf{C}^2$ such that $\langle s_i, s_j \rangle_A = \ell_{ij}$ on T.

Moreover, there exists a non-negative integer n such that

$$\mathcal{S}_{<n} = \emptyset \neq \mathcal{S}_n,$$

that is, there exists a selector that intersects the exterior section precisely n times; and all selectors intersect the exterior section at least n times.

The theorem is an immediate consequence of Theorem 3.5 below, where also the word "generically" will be made precise.

3.2. Reductions.

Lemma 3.2. – *$A \in \mathcal{L}$ if and only if there exists a global $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ frame field with respect to which A is given on T by the matrix-valued function*

$$\begin{bmatrix} |c|^2 - 1 & -\bar{c} \\ -c & 1 \end{bmatrix}$$

for some $c \in C^{k,\alpha}(T)$ with anti-holomorphic extension c into Δ satisfying $c(0) = 0$.

Proof. – Pick any global $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ frame field s_1, s_2; with respect to this frame field, $m_e = gs_1 + fs_2$, which we write $m_e \leftrightarrow [g\, f]^T$. Here the component functions belong to $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ and have no common zeros, Hence, by a version of the corona theorem, there exist $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$-functions F and G with

$$Ff - Gg \equiv 1.$$

With respect to the new $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ frame field $[\sigma_1\, \sigma_2]$ given by

$$[s_1\, s_2] = [\sigma_1\, \sigma_2] \begin{bmatrix} f & -g \\ -G & F \end{bmatrix},$$

$m_e \leftrightarrow [0\ 1]^T$; and since $\langle m_e, m_e \rangle_A > 0$ on T, in this frame we have (see [2], § 2.3)

$$A \leftrightarrow e^{\varphi} \begin{bmatrix} \bar{c}Bc - 1 & -\bar{c}B \\ -Bc & B \end{bmatrix}$$

on T for some $C^{k,\alpha}(T)$-functions $\varphi : T \to \mathbf{R}$, $c : T \to \mathbf{C}$, and $B : T \to \mathbf{R}_+$.

Picking a nonzero $\tilde{h} \in H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ with $|\tilde{h}|^2 = e^{\varphi}$ on T, with respect to the frame field $\tilde{h}[\sigma_1\, \sigma_2]$, the correspondence

$$A \leftrightarrow \begin{bmatrix} \bar{c}Bc - 1 & -B\bar{c} \\ -Bc & B \end{bmatrix}$$

holds on T. Since $B > 0$, in the same way we obtain a nonzero $h \in H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ with $|h|^2 = B$ on T, so with respect to the frame field

$$\tilde{h}[\sigma_1\, \sigma_2] \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}$$

we have

$$A \leftrightarrow \begin{bmatrix} |hc|^2 - 1 & -\bar{c}\bar{h} \\ -hc & 1 \end{bmatrix}$$

on T. Finally, we put $\gamma = hc$ and note that

$$\begin{bmatrix} |\gamma|^2 - 1 & -\bar{\gamma} \\ -\gamma & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\bar{\gamma} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix}$$

and that

$$\begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\gamma_a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\gamma_h & 1 \end{bmatrix},$$

where $\gamma = \gamma_h + \gamma_a$ according to the decomposition into parts having holomorphic and anti-holomorphic extensions into Δ (we have chosen constants so that $\gamma_a(0) = 0$). Since γ_h and γ_a belong to $C^{k,\alpha}(T)$, the frame field

$$\tilde{h}[\sigma_1\ \sigma_2] \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\gamma_h & 1 \end{bmatrix}$$

is one where the conclusion of the lemma holds with $c = \gamma_a$. ■

Remark 3.1. – In the above frame, we note that $m_e \leftrightarrow [0\,(h\tilde{h})^{-1}]^T$, so we might as well assume that m_e was chosen so that $m_e \leftrightarrow [0\ 1]^T$ in this frame. In other words, we think of the exterior section as the infinity section. □

By Lemma 3.2, in the sequel we may assume that A is a matrix-valued function given by

$$A = \begin{bmatrix} |c|^2 - 1 & -\bar{c} \\ -c & 1 \end{bmatrix} \tag{3.1}$$

for some $c \in C^{k,\alpha}(T)$ with anti-holomorphic extension satisfying $c(0) = 0$. We have to find a matrix-valued function $P \in GL(2, H(\Delta) \cap C^{k,\alpha}(\bar{\Delta}))$ such that

$$P^H A P = L \quad \text{on } T. \tag{3.2}$$

P is of course not uniquely determined: if P_1 and P_2 both satisfy (3.2), then, putting $M = P_1^{-1} P_2$,

$$M^H L M = L \quad \text{on } T, \tag{3.3}$$

that is, M is Lorentz-unitary on T, and the entries of M belong to $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$. But from (3.3) follows that they are also anti-holomorphic, so M is constant. Conversely, if P_1 solves (3.2) and $P_1^{-1}P_2$ is a constant Lorentz-unitary matrix, then P_2 solves (3.2).

It is straightforward to verify that M is Lorentz-unitary if and only if

$$M = \begin{bmatrix} a & b \\ \mu\bar{b} & \mu\bar{a} \end{bmatrix}$$

for some complex numbers a, b and μ satisfying $|a|^2 - |b|^2 = 1$ and $|\mu| = 1$; μ is then the determinant of M, $\det M$. Since

$$A = C^H LC, \quad \text{where } C = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix},$$

we thus need to find an invertible P with elements in $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ such that CP is Lorentz-unitary on T. In other words, we need to find a, b and μ in $C^{k,\alpha}(T)$ such that $|a|^2 - |b|^2 = 1$ and $|\mu| = 1$ on T and such that

$$P = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} a & b \\ \mu\bar{b} & \mu\bar{a} \end{bmatrix} = \begin{bmatrix} a & b \\ ac + \mu\bar{b} & bc + \mu\bar{a} \end{bmatrix}$$

has an invertible extension into Δ with elements in $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$; note that $\mu = \det P$ in that case.

3.3. Necessary conditions for existence. – If there is a solution P, then $\det P \in H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ is non-vanishing and $\det P = \mu$ on T. In particular, $|\det P| = 1$ on T, so $\det P$ is constant, and hence μ is.

3.4. Sufficient conditions for existence. – Suppose now, with no loss of generality, that $\mu = 1$ (in accordance with Subsection 3.3). We need to find a and b in $C^{k,\alpha}(T)$ such that all entries of the matrix

$$\begin{bmatrix} a & b \\ ac + \bar{b} & bc + \bar{a} \end{bmatrix} \tag{3.4}$$

have extensions in $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$, and such that the matrix extension is invertible everywhere in Δ.

Denote by Π, $\bar{\Pi} : C^{k,\alpha}(T) \to C^{k,\alpha}(T)$ the following projections, connected with the Hilbert transform,

$$\Pi \sum a_k z^k = \sum_{k \geq 1} a_k z^k, \tag{3.5a}$$

(3.5b) $$\bar{\Pi}\sum a_k z^k = \sum_{k\leq -1} a_k z^k,$$

where $z = e^{i\theta} \in T$. Of course, $a = \Pi(a) + a_0 + \bar{\Pi}(a)$ and $\bar{\Pi}(a) = \overline{\Pi(\bar{a})}$; and a has a holomorphic extension if and only if $\bar{\Pi}(a) = 0$. The entries of the matrix in (3.4) thus have holomorphic extensions if and only if

(3.6a) $$\Pi(a) = a - a_0,$$

(3.6b) $$\Pi(b) = b - b_0,$$

(3.6c) $$\Pi(b + \bar{a}\bar{c}) = 0,$$

(3.6d) $$\Pi(a + \bar{b}\bar{c}) = 0.$$

Since Π is a projection, this is equivalent to

(3.7a) $$a = a_0 - \Pi(\bar{b}\bar{c}),$$

(3.7b) $$b = b_0 - \Pi(\bar{a}\bar{c}),$$

or, eliminating b in (3.7a) and using that $\bar{\Pi}c = c$,

(3.8a) $$a - \Pi(\bar{c}\bar{\Pi}(ac)) = a_0 - \bar{b}_0\bar{c},$$

(3.8b) $$b = b_0 - \Pi(\bar{a}\bar{c}).$$

Introduce the operator $K_c : C^{k,\alpha}(T) \to C^{k,\alpha}(T)$ by

(3.9) $$K_c f = \Pi(\bar{c}\bar{\Pi}(cf))$$

(cf. the Hankel matrices in [1]). This operator is compact, since this is the case when c is a polynomial (only finitely many terms survive if we write $f = \sum f_k z^k$ and $c = \sum_N^M c_k z^k$), and an arbitrary $c \in C^{k,\alpha}(T)$ can be approximated in $C^{k,\alpha}(T)$-norm by polynomials. Now, (3.8a) can be written

$$(I - K_c)\, a = a_0 - \bar{b}_0 \bar{c}. \tag{3.10a}$$

If $I - K_c$ is invertible, a is uniquely determined from (3.10a), and so is b from (3.8b), and b is also the unique solution of

$$(I - K_c) b = b_0 - \bar{a}_0 \bar{c}. \tag{3.10b}$$

We note that equations (3.10a) and (3.10b) always have nontrivial solutions: either $I - K_c$ is invertible or has non-trivial kernel (and the choice $a_0 = 0 = b_0$ yields non-trivial solutions). However, we do not yet know if the resulting matrix is what we want. More precisely:

QUESTION. – *When do solutions a and b satisfy $|\,a\,|^2 - |\,b\,|^2 = 1$ on T?*

We investigate this by considering the operator-valued function

$$\mathbf{C} \ni \zeta \mapsto I - \zeta^2 K_c.$$

Since K_c is compact, the analytic Fredholm theorem (see, for instance, [6], § VI.5) implies that the set

$$D_c = \{\zeta \in \mathbf{C};\ I - \zeta^2 K_c \text{ is not invertible}\}$$

is discrete and does not contain zero. Moreover, $\zeta \mapsto (I - \zeta^2 K_c)^{-1}$ is meromorphic in $\mathbf{C}$, and analytic outside D_c, so we may define uniquely determined functions f^ζ and g^ζ through

$$f^\zeta = (I - \zeta^2 K_c)^{-1} 1, \tag{3.11a}$$

$$g^\zeta = (I - \zeta^2 K_c)^{-1} \zeta \bar{c}. \tag{3.11b}$$

For fixed $\zeta \notin D_c$, the functions belong to $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$; and they vary meromorphically with respect to $\zeta \in \mathbf{C}$.

When $\zeta \notin D_c$, the unique solutions a^ζ and b^ζ of

$$(I - \zeta^2 K_c) a^\zeta = a_0 - \zeta \bar{b}_0 \bar{c}, \tag{3.12a}$$

$$(I - \zeta^2 K_c) b^\zeta = b_0 - \zeta \bar{a}_0 \bar{c}, \tag{3.12b}$$

may thus be written

$$a^\zeta = a_0 f^\zeta - \bar{b}_0 g^\zeta, \tag{3.13a}$$

$$b^\zeta = b_0 f^\zeta - \bar{a}_0 g^\zeta, \tag{3.13b}$$

and hence

$$|a^\zeta|^2 - |b^\zeta|^2 = (|a_0|^2 - |b_0|^2)(|f^\zeta|^2 - |g^\zeta|^2). \tag{3.14}$$

We note that, for $\zeta \in \mathbf{R}\backslash D_c$, a^ζ and b^ζ are the solutions obtained by substituting ζc for c in (3.4). Moreover, for those ζ, the left-hand side of (3.14) is equal to the determinant of the corresponding P-matrix on T, and hence has a holomorphic extenstion. It is also real-valued on T, and so must be constant. Since we may choose a_0 and b_0 arbitrarily, and f^ζ and g^ζ are independent of those numbers, there is a continuous function $\psi : \mathbf{R}\backslash D_c \to \mathbf{R}$ such that

$$|f^\zeta(z)|^2 - |g^\zeta(z)|^2 = \psi(\zeta) \quad \text{for all } z \in T \text{ when } \zeta \in \mathbf{R}\backslash D_c. \tag{3.15}$$

When $\psi(\zeta) \neq 0$, we may choose a_0 and b_0 such that $|a^\zeta|^2 - |b^\zeta|^2 = 1$ on T and obtain a corresponding invertible P. The following lemma shows that the exceptional set where $\psi(\zeta) = 0$ is very small.

LEMMA 3.4. – *The set*

$$D'_c = \{\zeta \in \mathbf{R}\backslash D_c;\ \psi(\zeta) = 0\}$$

is a discrete subset of $\mathbf{R}$ *and does not contain zero.*

Proof. – Firstly, since $0 \notin D_c$ and $f^0 = 1$ and $g^0 = 0$, $\psi(0) = 1$, so $0 \notin D'_c$. Since D_c is discrete and ψ is continuous on $\mathbf{R}\backslash D_c$, D'_c avoids a real neighborhood of 0 as well.

Secondly, we show that the set of accumulation points of D'_c, acc D'_c, is open in $\mathbf{R}$; since it is trivially closed and does not contain 0, acc D'_c must then be empty, and the lemma follows. So, let $t_0 \in$ acc D'_c. Pick $z = 1$, say. Then $|f^\zeta(1)| = |g^\zeta(1)|$ when $\zeta \in D'_c$; and since $\zeta \mapsto f^\zeta(1)$ and $\zeta \mapsto g^\zeta(1)$ are meromorphic in ζ, in a complex neighborhood U of t_0, there is a way to form a holomorphic quotient $h(\zeta)$ of those functions;

either f^ζ/g^ζ or g^ζ/f^ζ will do (unless both of them are identically zero in this neighborhood, in which case t_0 is an interior point of acc D'_c, and we are done), and $|h(\zeta)| = 1$ on $U \cap (D'_c \cup \{t_0\})$, by continuity. We will show that $|h(\zeta)| = 1$ in a real neighborhood of t_0. To this end, let $H = \varphi \circ h$, where φ is a Möbius transformation taking the unit circle onto the real line and $h(t_0)$ to 0. When $\zeta = t$ is real and sufficiently close to t_0, we thus have

$$H(t) = \sum\nolimits_{k\geq 1} H_k (t - t_0)^k,$$

and we have to show that all H_k are real. Suppose H_N is the first nonreal coefficient; then

$$\frac{H(t) - \sum_{k=1}^{N-1} H_k (t-t_0)^k}{(t-t_0)^N} = H_N + O(|t - t_0|), \tag{3.16}$$

and we obtain a contradiction as $t \to t_0$ through $U \cap D'_c$, since the left-hand side of (3.16) is real for those values of t. We conclude that $|h(\zeta)| = 1$ in a real neighborhood of t_0, and hence t_0 is an interior point of acc D'_c. If follows that acc D'_c is open in $\mathbf{R}$, and the proof is complete. ∎

3.5. THE MAIN THEOREM. – Here we state the main theorem of this paper; note that Theorem 3.1 follows immediately from parts of this theorem.

THEOREM 3.5. – *Let*

$$A_c = \begin{bmatrix} |c|^2 - 1 & -\bar{c} \\ -c & 1 \end{bmatrix},$$

if $c \in C^{k,\alpha}(T)$; *and for non-negative integers* n, *let* $\mathcal{M}_n$ *be the set of functions that are meromorphic in* Δ, *belong to* $C^{k,\alpha}$ *in a neighborhood of* T, *and have precisely* n *poles in* Δ. *Introduce the meromorphic selectors*

$$\mathcal{S}_n^{tc} = \{m \in \mathcal{M}_n;\ |m(z) - tc(z)| \leq 1 \text{ on } T\}, \quad \text{and } \mathcal{S}_{<n}^{tc} = \cup_{k<n} \mathcal{S}_k^{tc},$$

and let

$$I_c = \mathbf{R} \backslash (D_c \cup D'_c),$$

where D_c *and* D'_c *are the exceptional sets defined above.*

Then the following holds:

1° $(D_c \cup D'_c) \cap \mathbf{R}$ *is a discrete subset of* $\mathbf{R}$ *and does not contain zero.*

2° *For every* $t \in I_c$, *there exists an everywhere invertible matrix-valued function* P_t *with elements in* $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ *such that* $P_t^H A_{tc} P_t = L$ *on* T. *Moreover, it is possible to choose* P_t *so that the map* $I_c \ni t \mapsto P_t$ *is real analytic.*

3° *There exists a locally constant function* $n : I_c \to \mathbf{N}$ *such that, for* $t \in I_c$,

$$\mathcal{S}^{tc}_{<n(t)} = \emptyset \neq \mathcal{S}^{tc}_{n(t)},$$

that is, there exists a meromorphic selector with precisely $n(t)$ *poles, and all selectors have at least* $n(t)$ *poles.*

4° *If we extend* A_{tc} *into* Δ *through* $A_{tc} = P_t^{-H} L P_t^{-1}$ *for* $t \in I_c$, *then*

$$\left\{(z, [w_0, w_1]) \in \bar{\Delta} \times \mathbf{P};\ [\bar{w}_0\ \bar{w}_1] A_{tc} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \leq 0\right\} = \bigcup_{m \in \mathcal{S}^{tc}_{n(t)}} \{(z, m(z)) \in \bar{\Delta} \times \mathbf{P}\}.$$

Moreover, the corresponding generalized Levi-flat surface is foliated by meromorphic disks with $n(t)$ *poles in* Δ.

5° *If* $\| c \|_\infty < 1$, *where* $\| \cdot \|_\infty$ *denotes the maximum norm on* T, *then there exists a Levi-flat surface with circular fibers*

$$S = \{(z, w) \in \bar{\Delta} \times \mathbf{C};\ |w - \gamma(z)| = r(z)\} \tag{3.17}$$

of class $C^{k,\alpha}(\bar{\Delta})$ *where* $\gamma = c$ *and* $r = 1$ *on* T.

Remark. – In the proof, we will identify meromorphic functions with nonzero holomorphic sections with values in $\mathbf{C}^2$, i.e.,

$$m \leftrightarrow \begin{bmatrix} g \\ f \end{bmatrix},$$

where $m = f/g$; of course, any two representatives f_1/g_1 and f_2/g_2 are related by $f_1 g_2 = f_2 g_1$. □

Proof of Theorem 3.5. – The first and second statements are clear from above.

As for the third statement, after a $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ change of frame we way assume that $A_{tc} = L$ and that m corresponds to f/g and ∞ to f_∞/g_∞ for functions in $H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ satisfying $f g_\infty \neq f_\infty g$ on T.

By hypothesis, $|f_\infty| > |g_\infty|$ on T. Suppose $|m(z) - tc(z)| \leq 1$ on T, i.e., $|f| \leq |g|$ on T. Then, in particular, $g \neq 0$ on T, so $|f_\infty g| > |fg_\infty|$ on T, and the number of poles of m in Δ is equal to

$$\frac{1}{2\pi}\,\Delta_T \arg(f_\infty g - fg_\infty) = \frac{1}{2\pi}\,\Delta_T \arg f_\infty\, g \geq \frac{1}{2\pi}\,\Delta_T \arg f_\infty,$$

where equality is attained when $g = 1$, for example. Note that

$$\begin{bmatrix} g_\infty \\ f_\infty \end{bmatrix} = P_t^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.18}$$

and since P_t^{-1} depends continuously on $t \in I_c$, so does f_∞. Hence the integer $\frac{1}{2\pi}\Delta_T \arg f_\infty$ is locally constant in t, and we define

$$n(t) = \frac{\Delta_T \arg f_\infty}{2\pi}.$$

The fourth statement is a consequence of the factorization $A_{tc} = P_t^{-H} L P_t^{-1}$: the given set is filled up with all meromorphic sections corresponding to

$$P_t \begin{bmatrix} 1 \\ f \end{bmatrix},$$

where $f \in H(\Delta) \cap C^{k,\alpha}(\bar{\Delta})$ and $|f| \leq 1$. From the proof of the third statement above, it is clear that the number of poles of all those meromorphic functions is one and the same, namely $n(t)$. The leaves of the foliation correspond to constant f with $|f| = 1$.

Finally, in order to prove the fifth statement, we infer that since $D_c \cup D'_c$ is discrete, for all $t \geq 0$ such that $t\|c\|_\infty < 1$ (except possibly for a finite set), there exist corresponding invertible P_t. Moreover, while the holomorphic zero section is inside the boundary circles, $|0 - tc(z)| < 1$ everywhere on T, from the third statement we conclude that $n(t) = 0$. The fourth one now yields a corresponding Levi-flat surface with circular fibers, and using the continuity-method result from [2], we see that such Levi-flat surfaces exist also when tc is replaced by sc if $0 \leq s \leq t$. Since $\|c\|_\infty < 1$, the above argument works for some $t > 1$, and thus a Levi-flat surface exists when $s = 1$. ■

Example. – Consider $c = \bar{z}^N$ for $N = 1, 2, 3, \ldots$ Writing $f = \sum f_k z^k$, it is straightforward to verify that

$$K_c f = \begin{cases} 0, & N = 1, \\ \sum_{k=1}^{N-1} f_k z^k, & N = 2, 3, \ldots \end{cases}$$

We see that $I - \zeta^2 K_c$ is injective (and hence invertible, by Fredholm theory) if $\zeta \neq \pm 1$; and when $\zeta = \pm 1$, the null space of $I - \zeta^2 K_c$ is spanned by $z, z^2, \ldots, z^{N-1}$. Hence

$$D_c = \begin{cases} \emptyset, & N = 1, \\ \{\pm 1\}, & N = 2, 3, \ldots \end{cases}$$

Furthermore, $f^t(z) = 1$ and $g^t(z) = tz^N$, and hence $\psi(t) = 1 - t^2$, so

$$D_c' = \begin{cases} \{\pm 1\}, & N = 1, \\ \emptyset, & N = 2, 3, \ldots \end{cases}$$

For all N, $I_c = \mathbf{R}\backslash\{\pm 1\}$; and when $|t| \neq 1$, we merely have to choose a_0 and b_0 such that

$$|a_0|^2 - |b_0|^2 = \frac{1}{1 - t^2}.$$

Suitable P_t are therefore

$$P_t = \begin{cases} \dfrac{1}{\sqrt{1 - t^2}} \begin{bmatrix} 1 & -tz^N \\ 0 & 1 - t^2 \end{bmatrix}, & |t| < 1, \\ \dfrac{1}{\sqrt{t^2 - 1}} \begin{bmatrix} -tz^N & 1 \\ t^2 - 1 & 0 \end{bmatrix}, & |t| > 1. \end{cases}$$

This gives us the uniquely determined extensions

$$A_{tc} = \begin{bmatrix} t^2 - 1 & -tz^N \\ -t\bar{z}^N & \dfrac{1 - t^2|z|^{2N}}{1 - t^2} \end{bmatrix}, \quad |z| \leq 1, \quad |t| \neq 1,$$

and the function $n(t)$ is given by

$$n(t) = \begin{cases} 0, & |t| < 1, \\ N, & |t| > 1. \end{cases}$$

When $N = 1$ and $|t| = 1$, from (3.14) we conclude that all solutions a and b satisfy $|a|^2 - |b|^2 = 0$ on T, and hence there is no P that is everywhere invertible in Δ. The best we can do is to allow P^{-1} to be singular at the origin, e.g.,

$$P_1 = \begin{bmatrix} z & 0 \\ 1 & 1 \end{bmatrix}.$$

Here, $P_1^H A_c P_1 = L$ on T, and

$$A_c = \frac{1}{|z|^2} \begin{bmatrix} 0 & -z \\ -\bar{z} & |z|^2 \end{bmatrix}, \quad z \neq 0,$$

is an unbounded vanishing-curvature extension that is singular at the origin. □

REFERENCES

[1] ADAMYAN (V. M.), AROV (D. Z.) and KREIN (M. G). – Infinite Hankel matrices and generalized problems of Carathéodory, Fejér and I. Schur, *Functional Analysis and Applications*, Vol. 2, 1968, p. 269-281.

[2] ALEXANDERSSON (L.). – On Vanishing-Curvature Extensions of Lorentzian Metrics, *Journal of Geometric Analysis*, Vol. 4, 1994, p. 425-466.

[3] BARRETT (D. E.) – Uniqueness for the Dirichlet Problem for Harmonic Maps from the Annulus into the Space of Planar Discs, *AMS Proceedings of Synposia in Pure Mathematics*, Vol. 52:1, 1991, p. 1-12.

[4] BERNDTSSON (B.). – Leviflat Surfaces with Circular Sections, *Several Complex Variables*, edited by J. Fornaess, p. 136-159. – Princeton, Princeton University Press, 1993 (*Proceedings of the Mittag-Leffler Institute*, 1988-1989).

[5] KOBAYASHI (S.). – *Differential Geometry of Complex Vector Bundles.* – Princeton, Princeton University Press, 1987.

[6] REED (M.) and SIMON (B.). – *Functional Analysis.* – Academic Press, 1980.

[7] WEGERT (E.). – *Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations.* – Berlin, Akademie Verlag, 1992 (*Mathematical Research*, Vol. 65).

[8] WIENER (N.) and MASANI (P.). – The Prediction Theory of Multivariate Stochastic Processes, *Acta Mathematica*, Vol. 98, 1957, p. 111-150.